arXiv:1110.2632v1 [math-ph] 12 Oct 2011

Deformation quantization of rank I Bergman domain

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LPT-2011-30

Abstract

In this paper we present a harmonic oscillator realization of the most degenerate discrete series representations of the $SU(m,1)$ group and the deformation quantization of the coset space $D = SU(m,1)/U(m)$ with the method of coherent state quantization. We study in more detail the case of $m = 2$ and give an explicit expression of the star product. Finally we consider a quantum field theory model over the quantized domain $\hat{D}$.

1 Introduction

This paper is a continuation of a previous one [1] where we considered the deformation quantization of the coset space $SU(2,2)/S(U(2) \times U(2))$. In the current paper we consider the deformation quantization of the rank 1 Hermitian symmetric spaces, namely the coset space $SU(m,1)/S(U(m) \times U(1))$. Geometrically this is the m dimensional complex ball [2] with negative constant curvature. We study more carefully the deformation quantization of the domain $SU(2,1)/U(2)$, since the $m \geqslant 3$ case is not more complicated than $m = 2$ case. We build also a quantum field theory model on this deformed domain. Since the deformed coset space has real dimension 4, we expect that a QFT model on this domain could have some physically interest properties.

This paper is organized as follows. In Section 2 we give an general introduction to the noncompact unitary group $G = SU(m,1)$, its Lie algebra $\mathbf{g} = su(m,1)$ and the corresponding maximal symmetric space $SU(m,1)/U(m)$. In section 3 ,we present a simple harmonic oscillator realization of the most degenerate discrete series of representations of $SU(m,1)$ group which generalizes the more common (Schwinger-Jordan) oscillator realizations used in the case of compact groups. And in appendix B we generalize this method to an arbitrary $SU(m,n)$ group. The relevant mathematical background of group theory and representation theory can be found in, e.g.,[3, 4, 5, 6].

In Section 4 we construct the system of coherent states [7] for the representation in question and give a corresponding star-product formula for the noncommutative algebra of functions defined on the Bergman domain $D = SU(2,1)/U(2)$. The non-commutative algebra shall then induce a noncommutative structure on the domain D. This is the basic principle of the coherent state quantization method [8].

The construction of the coset space $SU(2,1)/U(2)$ has been also studied by [9], [10] with the method of Berezin-Toeplitz quantization and by by [11] with the method of "WKB quantization". The interested reader could go to the references for details.

In Section 5 we consider a scalar field theory model on the quantized Bergman domain D and calculate the amplitude of the tadpole graph.

Finally in appendix A we give the explicit form of the killing vectors of the $SU(2,1)$ group and in appendix C we give the explicit form of the invariant Laplacian operator for the domain D.

2 The $SU(m,1)$ group and its Lie algebra

2.1 Brief introduction to the group $SU(m,n)$

The group $G = SU(m,n)$, $m \geq n > 0$, is defined as a subgroup of the matrix group $SL(m+n,C)$: taking the corresponding (m,n) partitions of rows and columns, any matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \tag{1}$$

satisfies the constraint

$$g^\dagger \Gamma g = g \Gamma g^\dagger = \Gamma, \quad \Gamma = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -I_{n\times n} \end{pmatrix}. \tag{2}$$

Here I's represents unit matrices and 0's the blocks of zeros.

From equation (2) follow two equivalent sets of constraints

$$a^\dagger a = I + c^\dagger c, \quad d^\dagger d = I + b^\dagger b, \; a^\dagger b = c^\dagger d \tag{3}$$

or

$$aa^\dagger = I + bb^\dagger, \; dd^\dagger = I + cc^\dagger, \; ac^\dagger = bd^\dagger. \tag{4}$$

We could easily find that the matrices a and d are invertible.

The maximal compact subgroup $K = S(U(m) \times U(n))$ of G is defined by matrices

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K, \quad k_1, k_2 \; - \; \text{unitary}, \quad \det(k_1\, k_2) = 1. \tag{5}$$

The transformations generated by $g \in G$ in $\mathbf{C}^{m+n}$: $\zeta \mapsto g\zeta$, leave invariant quadratic form:

$$\zeta^\dagger \Gamma \zeta = |\zeta_1|^2 + \cdots + |\zeta_m|^2 - |\zeta_{m+1}|^2 - \cdots - |\zeta_{m+n}|^2.$$

The Lie algebra $\mathbf{g} = su(m,n)$ of G is defined by matrices X satisfying the condition

$$X^\dagger \Gamma \; + \; \Gamma X \; = \; 0, \tag{6}$$

So that every element $X \in \mathbf{g}$ has the form:

$$X \; = \; \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}, \tag{7}$$

such that

$$A^\dagger = -A,\ D^\dagger = -D, \quad \mathrm{tr}(A) + \mathrm{tr}(D) \,=\, 0,$$

and B is an $m \times n$ complex matrix. The Cartan decomposition of $\mathbf{g}$ reads (see [4][3][17]):

$$\mathbf{g} = \mathbf{k} + \mathbf{p}, \tag{8}$$

where the set $\mathbf{p}$ contains matrices of the form $\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}$ and $\mathbf{k}$ means the compact part $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$.

The mappings in $\mathbf{C}^{m+n}$ generated by compact generators from $\mathbf{k}$ we call *rotations*, the mappings generated by by noncompact generators from $\mathbf{p}$ we call *boosts*.

Let $\mathbf{a} \in \mathbf{p}$ be a maximal abelian subalgebra in $\mathbf{p}$. We may choose, in the $(m-n, n, n)$ partition of rows and columns, for $\mathbf{a}$ the set of all real matrices of the form:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Lambda \\ 0 & \Lambda & 0 \end{pmatrix}, \quad \Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n). \tag{9}$$

The corresponding abelian group $\delta\Lambda$ is obtained by exponential map and contains matrices of the form:

$$\delta_\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C & S \\ 0 & S & C \end{pmatrix}, \quad C = \mathrm{ch}\,\Lambda, \quad S = \mathrm{sh}\,\Lambda. \tag{10}$$

The principal Cartan subgroup H of G is the maximal commutative subgroup of G containing A as subgroup: $H = M \times A$: M and A are the subgroups that enter the Iwasawa decomposition of the minimal parabolic subgroup of G. The Lie algebra $\mathbf{a}$ of the Cartan subgroup H has the form

$$\mathbf{h} = \mathbf{m} \oplus \mathbf{a},$$

where $\mathbf{m}$ contains all matrices from $\mathbf{k}$ commuting with $\mathbf{a}$. Explicitly, $\mathbf{m}$ contains traceless matrices of the form:

$$X = \begin{pmatrix} D & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \begin{matrix} D^\dagger = -D \text{ diagonal} \\ R^\dagger = -R \text{ arbitrary} \end{matrix} \tag{11}$$

Any element of G possesses a unique $K\delta K$ decomposition [12]:

$$g \;=\; k\,\delta\,\tilde{q} \;=\; \tilde{k}\,\delta\,q\,, \tag{12}$$

where δ is an element of A specified by positive parameters $\lambda_1, \ldots, \lambda_n$ in formula (10). Further, $k = \tilde{k}\,u$ and $q = u\,\tilde{q}$ are elements of K, where $u \in M$ and $\tilde{k} \in K/M$, $\tilde{q} \in M \setminus K$.

2.2 The $SU(m,1)$ group

The definitions of the group $SU(m,1)$ and its corresponding Lie algebra follow the $SU(m,n)$ case by setting $n=1$.

Consider the Cartan decomposition of the Lie algebra $\mathbf{g}=\mathbf{l}+\mathbf{p}$ and let $\mathbf{a}\in\mathbf{p}$ be a maximal abelian subalgebra. We could choose for $\mathbf{a}$ the set of all matrices of the form

$$H_t=\begin{pmatrix} O_{(m-1)\times(m-1)} & O_{(m-1)\times 1} & O_{(m-1)\times 1} \\ O_{1\times(m-1)} & 0 & t \\ O_{1\times(m-1)} & t & 0 \end{pmatrix} \tag{13}$$

where t is a real number.

Define the linear functional over H_t by $\alpha(H_t)=t$, the roots of $(\mathbf{g},\mathbf{a})$ are given by

$$\pm\alpha,\ \pm 2\alpha, \tag{14}$$

with multiphicities $m_\alpha=2$ and $m_{2\alpha}=1$.

Define

$$\delta:=\{\mathbf{a}_t|\ \mathbf{a}_t=\exp H_t,\ H_t\in\mathbf{a}\}. \tag{15}$$

so we have

$$\mathbf{a}_t=\begin{pmatrix} I & O & 0 \\ O & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \tag{16}$$

where the symbol I stands for the identity matrix and O is the matrix with entries 0 .

On the root system we choose that the positive Weyl chamber given by $C^+=\{t\}$ with $t>O$. Then the positive roots are α and 2α. And the simple root is α.

Now we consider the $K\delta K$ decomposition of an arbitrary group element $g\in G$:

$$g=k\delta q^\dagger, \tag{17}$$

where $k,\ q\in S(U(m)\times U(1))$.

We could write the Haar measure of group g as:

$$dg=dg(t,k,q)=\rho(t)dtdkdq, \tag{18}$$

where dk and dq are normalized Haar measure on the maximal compact subgroup $U(m)=S(U(m)\times U(1))$, and $\rho(t)dt$ is the measure on the noncompact group element. The explicit form of $\rho(t)$ could be derived from the positive roots, see ([4]):

$$\rho(t)=\prod_{\alpha\in\Sigma^+}|\sinh\alpha(t)|^{m_\alpha}, \tag{19}$$

where m_α is the multiplicity of the positive roots.

So we have:

$$\rho(t)=\sinh^2 t\sinh 2t. \tag{20}$$

2.3 The maximal symmetric space of $SU(m,1)$

The corresponding maximal symmetric space (also called the the Bergman domain) is defined as the coset of the group $SU(m,1)$ over its maximal compact subgroup $S(U(m) \times U(1))$.

More precisely, for each element $g \in SU(m,1)$ we have the following Cartan decomposition:

$$g = \begin{pmatrix} N_1 & ZN_2 \\ Z^\dagger N_1 & N_2 \end{pmatrix} \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \tag{21}$$

where $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \in S(U(m) \times U(1))$ is an element of the maximal subgroup, $N_1 = (E - ZZ^\dagger)^{-1/2}$, $N_2 = (1 - Z^\dagger Z)^{-1/2}$, and

$$Z = bd^{-1} = \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_m \end{pmatrix}, \;\; Z^\dagger = (z_1^\dagger, z_2^\dagger, \cdots z_m^\dagger). \tag{22}$$

The Bergman domain D is defined by :

$$D = \{Z \,|\, 1 - |Z|^2 > 0\} \;\; = \;\; \{z \,|\, 1 - |z_1|^2 - |z_2| - \cdots - |z_m|^2 > 0\}. \tag{23}$$

It is a pseudo-convex domain over which we could define a holomorphic Hilbert spaces [9] with the reproducing Bergman kernel:

$$K(W^\dagger, Z) = (1 - W^\dagger Z)^{-N}, \tag{24}$$

where Z and W are complex m-columns, and $N = m+1, m+2, \cdots$ is a natural number characterizing the representation.

What's more, the Bergman domain D is a Kähler manifold with the Kähler metric defined by the derivations of the Bergman kernel:

$$g_{i\bar{j}} \;\; = \frac{1}{N} \partial_{\bar{z}^i} \partial_{z^j} \; \log K(Z^\dagger, Z). \tag{25}$$

More explicitly we have:

$$g_{i\bar{j}} = [\frac{\delta_{ij}}{1 - |Z|^2} + \frac{z_i \bar{z}_j}{(1 - |Z|^2)^2}], \;\; g^{i\bar{j}} = (1 - |Z|^2)(\delta_{ij} - \bar{z}_i z_j). \tag{26}$$

The Christoffel symbols can be easily derived:

$$\Gamma^{\bar{l}}_{\bar{j}\bar{k}} \;\; = g^{s\bar{l}} \partial_{\bar{z}_j} g_{s\bar{k}} \;\; = \frac{\delta_{kl} z_j + \delta_{jl} z_k}{1 - |Z|^2}, \tag{27}$$

the Ricci tensor reads:

$$R_{i\bar{j}} = -\partial_{z_i} \Gamma^{\bar{k}}_{\bar{j}\bar{k}} \;\; = -(d+1)[\frac{\delta_{ij}}{1 - |Z|^2} + \frac{\bar{z}_i z_j}{(1 - |Z|^2)^2}] = -(d+1) g_{i\bar{j}}, \tag{28}$$

and the scalar curvature reads:

$$R = g^{i\bar{j}} R_{i\bar{j}} = -(m+1), \tag{29}$$

where m is the complex dimension of D.

We could easily verify that the metric $g_{i\bar{j}}$ is a solution to the Einstein's equation in the vacuum:

$$R_{i\bar{j}} - \frac{1}{2} g_{i\bar{j}} R + \Lambda g_{i\bar{j}} = 0 \tag{30}$$

and the cosmological constant reads

$$\Lambda = \frac{m+1}{2} \tag{31}$$

The coset space D is also called the Kähler-Einstein manifold and is well known by the mathematicians [13].

D is not simply connected but has genus $m+1$. The simplest way of calculating the genus of an arbitrary complex ball is by calculating the corresponding complex Jordan triple, see [9, 17] for more details.

3 The holomorphic discrete series of representations of the $SU(m,1)$ group

The noncompact group $SU(m,1)$ possesses three kinds of unitary irreducible representations: the principal series, the complementary series and the discrete series. In section (3.1) we will give a harmonic oscillator realization of the discrete series of representation of the $SU(m,1)$ group. This method is a generalization of the one in [1]. As the $m \geq 3$ case is quite similar to the $m=2$ case, we shall take $m=2$ in most part of this subsection. Then in appendix B we generalize this method to the $SU(m,n)$ case.

3.1 The harmonic oscillator realization of the discrete series of representation of $SU(m,1)$ group

The Group $G = SU(m,1)$ possesses three unitary irreducible representations: the principal series, the discrete series and the supplementary series. We consider only the discrete series of representations in this paper.

The discrete series of representations is realized in the Hilbert space $\mathcal{L}^2_N(D)$ of holomorphic functions in the Bergman domain with the invariant measure $d\mu_N(Z,\bar{Z})$ given by:

$$d\mu_N(Z,\bar{Z}) = c_N [\det(E - Z^\dagger Z)]^{N-(m+1)} |dZ| \tag{32}$$

where $|dZ|$ is the Lebesgue measure in C^m.

We could choose the normalization constant c_m to be $c_N = \pi^{-2}(N-2)(N-1)$ so that

$$\int d\mu_N(Z,\bar{Z}) = 1. \tag{33}$$

Then in the space $\mathcal{L}^2_N(D)$ we define the discrete series of representations T by

$$T_N f(Z) = [\det(CZ+d)]^{-N} f(Z'),\ \ N = m+1, m+2, \cdots \tag{34}$$

where

$$Z' = (AZ+B)(CZ+d)^{-1} \tag{35}$$

In the following we consider $m = 2$ case, as in this case the Bergman domain $D = SU(2,1)/U(2)$ has real dimension 4 and might be more interesting for real physical system. Remark that our method in the follows could be easily generalized to arbitrary m.

Below we construct an oscillator realization of most degenerate discrete series representations depending on one natural number N [1].

We introduce a 3×1 matrix $\hat{Z} = (\hat{z}_a)$, $a = 1,2,3$, of bosonic oscillators acting in Fock space and satisfying commutation relations

$$[\hat{z}_a, \hat{z}_b^\dagger] = \Gamma_{ab},\ \ a,b = 1,2,3 \tag{36}$$

$$[\hat{z}_a, \hat{z}_b] = [\hat{z}_a^\dagger, \hat{z}_b^\dagger] = 0, \tag{37}$$

where Γ is a 3×3 matrix defined in (2). It can be easily seen that for all $g \in SU(2,1)$ these commutation relations are invariant under transformations:

$$\hat{Z} \ \mapsto \ g\,\hat{Z}, \quad \hat{Z}^\dagger \ \mapsto \ \hat{Z}^\dagger\, g^\dagger. \tag{38}$$

Since, $\Gamma = \operatorname{diag}(+1,+1,-1)$ the upper two rows in $\hat{Z}$ corresponds to annihilation operators whereas the lower one to creation operators:

$$\hat{Z} \ = \ \left(\begin{array}{c} \hat{a} \\ \hat{b}^\dagger \end{array}\right):\ \ [\hat{a}_\alpha, \hat{a}_\beta^\dagger] \ = \ \delta_{\alpha\beta}, \alpha, \beta = 1,2.\ \ [\hat{b}, \hat{b}^\dagger] \ = \ 1, \tag{39}$$

and all other commutation relations among oscillator operators vanish. Here $\hat{a}$ represents a column of two oscillators $\hat{a}_1$ and $\hat{a}_2$.

The Fock space $\mathcal{F}$ in question is generated from a normalized vacuum state $|0\rangle$, satisfying $\hat{a}_\alpha\,|0\rangle \ = \ \hat{b}\,|0\rangle \ = \ 0$, by repeated actions of creation operators:

$$|m_\alpha,\, n\rangle \ = \ \prod_\alpha \frac{(\hat{a}_\alpha^\dagger)^{m_\alpha}\,(\hat{b}^\dagger)^n}{\sqrt{m_\alpha!\, n!}}\,|0\rangle\,. \tag{40}$$

We shall use the terminology that the state $|m_\alpha,\, n\rangle$ contains $M = \sum m_\alpha$ particles a and n particles b.

The Lie algebra $su(2,1)$ acting in Fock space can be realized in terms of oscillators. Consider a basis of $su(2,1)$ Lie algebra $X \ = \ X^A_{ab}$ $A = 1,\cdots 8$, $a,b = 1,2,3$, we assign the operator

$$\hat{X} \ = \ -\hat{Z}^\dagger \Gamma X \hat{Z} \ = \ -\hat{z}_a^\dagger\, \Gamma^A_{ab}\, X^A_{ab}\, \hat{z}_b\,, \tag{41}$$

with $\hat{Z}^\dagger$ and $\hat{Z}$ given in (39). Their anti-hermicity follows directly:

$$\hat{X}^\dagger \ = \ -\mathrm{tr}\,(\hat{Z}^\dagger X^\dagger \Gamma \hat{Z}) \ = \ +\,\mathrm{tr}\,(\hat{Z}^\dagger \Gamma X \hat{Z}) \ = \ -\,\hat{X}.$$

Using commutation relations for annihilation and creation operators the commutator of operators $\hat{X} = \hat{Z}^\dagger \Gamma X \hat{Z}$ and $\hat{Y} = \hat{Z}^\dagger \Gamma Y \hat{Z}$ can be easily calculated:

$$[\hat{X}, \hat{Y}] = [\hat{Z}^\dagger \Gamma X \hat{Z}, \hat{Z}^\dagger \Gamma Y \hat{Z}] = -\hat{Z}^\dagger \, \Gamma \, [X, \ Y] \, \hat{Z} \,. \tag{42}$$

It can be easily seen that the anti-hermitian operators $\hat{X}_a$, satisfy in Fock space the $su(2,1)$ commutation relations. The assignment

$$g = e^{\xi^A X_A} \in SU(2,1) \Rightarrow \hat{T}(g) = e^{\xi^A \hat{X}_A} \tag{43}$$

then defines a unitary $SU(2,1)$ representation in Fock space. The explicit form of the operators X_a is shown in the appendix.

The adjoint action of $\hat{T}(g)$ on operators reproduces (92). In terms of a and b-oscillators in block-matrix notation this can be rewritten as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \begin{array}{ll} \hat{T}(g) \, \hat{a} \, \hat{T}^\dagger(g) = a \, \hat{a} + b \, \hat{b}^\dagger, & \hat{T}(g) \, \hat{a}^\dagger \hat{T}^\dagger(g) = \hat{a}^\dagger \, a^\dagger + \hat{b} \, b^\dagger, \\ \hat{T}(g) \, \hat{b}^\dagger \, \hat{T}^\dagger(g) = d \, \hat{b}^\dagger + c \, \hat{a}, & \hat{T}(g) \, \hat{b} \, \hat{T}^\dagger(g) = \hat{b} \, d^* + \hat{a}^\dagger \, c^\dagger, \end{array} \tag{44}$$

where $\star$ means the complex conjugation and $\dagger$ means either the complex-transpose of the matrices or the hermitian conjugation or the operators in Fock space. Since any $g \in SU(2,1)$ possesses the Cartan decomposition (17) we shall discuss separately rotations and special boosts given in (10). For rotations we obtain a mixing of annihilation and creation operators of a same type:

$$k = \begin{pmatrix} k' & 0 \\ 0 & k'' \end{pmatrix} : \quad \begin{array}{ll} \hat{T}(k) \, \hat{a} \, \hat{T}^\dagger(k) = k' \, \hat{a}, & \hat{T}(k) \, \hat{a}^\dagger \hat{T}^\dagger(k) = \hat{a}^\dagger \, k'^\dagger, \\ \hat{T}(k) \, \hat{b}^\dagger \, \hat{T}^\dagger(k) = k'' \, \hat{b}^\dagger, & \hat{T}(k) \, \hat{b} \, \hat{T}^\dagger(k) = \hat{b} \, k''^\dagger. \end{array} \tag{45}$$

where $k' \in U(2)$, $k'' \in U(1)$ and $\det(k'k'') = 1$.

However, for the noncompact elements we obtain Bogolyubov transformations for the oscillators $\hat{a}$ and b. For example, for the transformation

$$\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} , \tag{46}$$

generated by X_7 given in the appendix we have the Bogolyubov transformations for for $\hat{a}_2$ and b:

$$\begin{array}{rcl} \hat{T}(\delta) \, \hat{a}_1 \, \hat{T}^\dagger(\delta) & = & \hat{a}_1, \quad \hat{T}(\delta) \, \hat{a}_2 \hat{T}^\dagger(\delta) = \cosh t \hat{a}_2 + \sinh t \hat{b}^\dagger, \\ \hat{T}(\delta) \, \hat{b}^\dagger \, \hat{T}^\dagger(\delta) & = & \sinh t \hat{a}_2 + \cosh t \hat{b}^\dagger. \end{array} \tag{47}$$

Remark that $\hat{a}_1$ doesn't change under this Bogolyubov transformation.

We could easily get the Bogolyubov transformations for $a_2^\dagger$ and b by taking the Hermitian conjugation of the above formula.

Using the explicit form of matrices X^A, which will be given in the appendix, following from (44), the action of generators can be described in terms creation and annihilation of a- and b-particles:

(i) The action of *rotation* generators results in a replacement of some $a_{1,2}$-particle by an other $a_{1,2}$-particle and by replacement of b-particle by other b-particle.

(ii) The action of *boost* generators results in creation of certain pair of $(a_\alpha b)$ of particles or in a destruction of $a_\alpha b$ pair, where $\alpha = 1, 2$.

In this context it is useful to consider lowering and rising operators that annihilate and create $a_\alpha b$ pairs:

$$T_- \;=\; \hat{a}_\alpha \, \hat{b}, \quad T_+ = (T_-)^\dagger = \; \hat{b}^\dagger \hat{a}^\dagger_\alpha, \tag{48}$$

So that any boost given by (41) of a complex combination of X_A, $A = 4, 5, 6, 7$, can be uniquely expressed as complex combinations of operators T_- and T_+.

It follows from (48) that the operator

$$\hat{N} \equiv \; \hat{N}_{\hat{b}} - \hat{N}_{\hat{a}} \;=\; \hat{b}^\dagger \hat{b} - \hat{a}^\dagger_1 \hat{a}_1 - \hat{a}^\dagger_2 \hat{a}_2 = -[\hat{Z}^\dagger \Gamma \hat{Z} + 1] \tag{49}$$

commutes with all generators $\hat{X}^A$.

Below, we shall restrict ourselves to most degenerate discrete series representations which are specified by the eigenvalue of the operator $\hat{N}$ in the representation subspace. We start to construct the representation space $\mathcal{F}_N$ from a distinguished normalized state containing lowest number of particles:

$$|x_0\rangle = \frac{(\hat{b}^\dagger)^N}{\sqrt{(N)!}} |0>= \frac{1}{\sqrt{N}} \, |0, 0; N\rangle \, . \tag{50}$$

Here N is a natural number that specifies the representation: $\hat{N} \, |x_0\rangle \;=\; N \, |x_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in (48): such states contain besides N b-particles a finite number of ab pairs.

The maximal compact subgroup $K \;=\; S(U(2) \times U(1))$ is the stability group of the state $|x_0\rangle$. The group action for $k \;=\; \mathrm{diag}(k', k'')$ reduces just to the phase transformation (see (38) or (44)):

$$\hat{b}^\dagger \;\mapsto\; \hat{b}^\dagger \, k'' \quad \Rightarrow \quad |x_0\rangle \;\mapsto\; e^{iN\alpha} \, |x_0\rangle \tag{51}$$

where $k'' = e^{i\alpha(k)}$ is the $U(1)$ part of K.

Let us calculate the mean values of the operator $\hat{T}(g)$ in the state $|x_0\rangle$: $\omega_0(g) = \langle x_0 | \, \hat{T}(g) \, |x_0\rangle$.

Using the decomposition $g = k^\dagger \, \delta \, q$ and the action of rotations (51) we obtain:

$$\begin{aligned} \omega_0(g) \;&=\; \langle x_0 | \, \hat{T}(g) \, |x_0\rangle \;=\; \langle x_0 | \, \hat{T}(k)^\dagger \, \hat{T}(\delta) \, \hat{T}(q) \, |x_0\rangle \\ &=\; e^{iN(\alpha(q) - \alpha(k))} \langle x_0 | \, \hat{T}(\delta) \, |x_0\rangle \end{aligned} \tag{52}$$

Thus it is enough to calculate the mean value for a noncompact element:

$$\begin{aligned} \delta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} = t_+ \; t_0 \; t_- \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \tanh t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh^{-1} t & 0 \\ 0 & 0 & \cosh t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \tanh t & 1 \end{pmatrix} \end{aligned} \tag{53}$$

In the representation in question, the matrices t_+ and t_- are the exponents of the matrices

$$X_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tanh t \\ 0 & 0 & 0 \end{pmatrix}, \ X_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tanh t & 0 \end{pmatrix}, \tag{54}$$

respectively. Therefore we have

$$\hat{T}(t_+) \ = \ e^{\hat{a}_2^\dagger \, \tanh t \, \hat{b}^\dagger}, \ \ \hat{T}(t_-) \ = \ e^{b \, \tanh t \, a_2}. \tag{55}$$

Since $\hat{T}(t_-)$ contains $\hat{a}_2$ its action does not affect $|x_0\rangle$, and similarly $\hat{T}(t_+)$ containing $\hat{a_2}^\dagger$ does not affect $\langle x_0|$. The only non-trivial action comes from

$$\hat{T}(t_0) \ = \ e^{-b\hat{b}^\dagger \ln \cosh t + \hat{a_2}\hat{a_2}^\dagger \ln \cosh t} \ = \ \frac{1}{\cosh t} e^{-\hat{b}^\dagger b \ln \cosh t + \hat{a_2}\hat{a_2}^\dagger \ln \cosh t}. \tag{56}$$

Since the actions of the operators $\hat{a}_2$ and $\hat{a_2}^\dagger$ on the state $|x_0\rangle$ are trivial, we have

$$\hat{T}(t_0)|x_0\rangle = \frac{1}{\cosh t} e^{-\hat{b}^\dagger b \ln \cosh t}|x_0\rangle \tag{57}$$

$$= \ \frac{1}{\cosh t}\sum_{n=0}^{N}(-1)^n \frac{(\ln \frac{1}{\cosh t})^n}{n!}(b^\dagger)^n b^n \frac{(b^\dagger)^N}{\sqrt{(N)!}}|0\rangle$$

$$= \ \frac{1}{\cosh t}\sum_{n=0}^{N}(-1)^n C_N^n (\ln \frac{1}{\cosh t})^n \frac{(b^\dagger)^N}{\sqrt{(N)!}}|0\rangle = \frac{1}{\cosh t}(1+\ln\cosh t)^N |x_0\rangle$$

From equations (51) and (56) we obtain:

$$\omega_0(g) = \langle x_0|\,\hat{T}(g)\,|x_0\rangle = \frac{1}{\cosh t}[(1+\ln\cosh t)e^{i(\alpha(q)-\alpha(k))}]^N \tag{58}$$

.

4 The Coherent States Quantization of $D = SU(2,1)/U(2)$ and the Star Product

Starting from the normalized state $|x_0\rangle \in \mathcal{F}_N$ we shall construct the Perlomov's system of coherent states for the representation in question. We choose a set of boosts of the form:

$$g_x = k\,\delta\,k^\dagger \ \ = \begin{pmatrix} C' & S' \\ S'^\dagger & C'' \end{pmatrix} \in G/K, \tag{59}$$

where $C' = k' \begin{pmatrix} 1 & 0 \\ 0 & \cosh t \end{pmatrix} k'^\dagger$, $S' = \begin{pmatrix} k'_{12} \sinh t \\ k''_{22} \sinh t \end{pmatrix} k''^\star$, $C'' = \cosh t$. k' and k'' are defined by formula (45). To any boost g_x defined in (59), we assign coherent coherent state (see, [7]):

$$|x\rangle \ = \ \hat{T}(g_x)\,|x_0\rangle \ = \ \hat{T}(k\,\delta\,k^\dagger)\,|x_0\rangle, \ \ x = x(k,\delta). \tag{60}$$

Let us consider operators in the representation space of the form

$$\hat{F} \;=\; \int_G dg\, \tilde{F}(g)\, \hat{T}(g), \tag{61}$$

where $\tilde{F}(g)$ is a distribution on a group G with compact support. To any such operator we assign function on G/K by the prescription

$$F(x) \;=\; \langle x|\, \hat{F}\, |x\rangle \;=\; \int_G dg\, \tilde{F}(g)\, \omega(g,x), \tag{62}$$

where

$$\omega(g,x) \;\equiv\; \langle x|\, \hat{T}(g)\, |x\rangle \;=\; \omega_0(g_x^{-1}\, g\, g_x). \tag{63}$$

This equation combined with (58) offers an explicit form of $\omega(g,x)$ and is well suited for calculations.

The star-product of two functions $F(x) = \langle x|\, \hat{F}\, |x\rangle$ and $G(x) = \langle x|\, \hat{G}\, |x\rangle$ was defined by [8]:

$$\begin{aligned} (F \star G)(x) \;&=\; \langle x|\, \hat{F}\hat{G}\, |x\rangle \;=\; \int_{G\times G} dg_1 dg_2\, \tilde{F}(g_1)\, \tilde{G}(g_2)\, \omega(g_1 g_2, x) \\ &=\; \int_G dg\, (\tilde{F} \circ \tilde{G})(g)\, \omega(g,x), \end{aligned} \tag{64}$$

where the symbol $\tilde{F} \circ \tilde{G}$ denotes the convolution in the group algebra $\tilde{\mathcal{A}}_G$ of compact distributions:

$$(\tilde{F} \circ \tilde{G})(g) \;=\; \int_G dh\, \tilde{F}(gh^{-1})\, \tilde{G}(h), \tag{65}$$

and it can be easily seen that

$$\operatorname{supp}(\tilde{F} \circ \tilde{G}) \;\subset\; (\operatorname{supp} \tilde{F})\, (\operatorname{supp} \tilde{G})$$

$$\equiv\; \{g \;=\; g_1 g_2 \,|\, g_1 \in \operatorname{supp} \tilde{F},\; g_2 \in \operatorname{supp} \tilde{G}\}.$$

Obviously the star product defined above is associative and is invariant under the action of the group $SU(2,1)$.

Consequently, for a non-compact group there are two classes of group algebras:

(i) The first one is generated by distributions with a general compact support and the corresponding group algebra is simply the full algebra $\tilde{\mathcal{A}}_G$ defined in (64).

(ii) The second one is formed by distributions $\tilde{F}$ with $\operatorname{supp} \tilde{F}$ subset of a subgroup $H \subset K$, form a sub-algebra $\tilde{\mathcal{A}}_H$ of the group algebra $\tilde{\mathcal{A}}_G$.

Note: We point out that as in the case of usual distributions, the convolution product may exist even for distribution with non-compact support provided there are satisfied specific restriction at infinity.

In the second class there are two interesting extremal cases:

(a) $\tilde{\mathcal{A}}_{\{e\}}$ corresponding to the trivial subgroup $H = \{e\}$ in $G = SU(2,1)$.

(b) $\tilde{\mathcal{A}}_K$ corresponding to the maximal compact subgroup K in G.

Note: It is well-known that $\tilde{\mathcal{A}}_{\{e\}}$ is isomorphic to the enveloping algebra $\mathcal{U}(su(2,1))$ (see e.g., [4]).

Obviously, the mapping given in (62), assigning to any compact distribution $\tilde{F}$ the function F, is a homomorphism of the group algebra $\tilde{\mathcal{A}}_G$ into the star-algebra $\mathcal{A}^\star_G$ of functions (62) on G/K.

The deformation quantization on Lie group co-orbits in terms of the Lie group convolution algebra $\tilde{\mathcal{A}}_{\{e\}}$ was introduced by [14]. Here we follow the related coherent state construction of the star-star on Lie group co-orbits proposed in [8].

Any distribution $\tilde{F}$ can be given as a linear combination of finite derivatives of the group δ-function, i.e., as a linear combination of distributions

$$\tilde{F}_{A_1\ldots A_n}(g) \; = \; (\mathcal{X}_{A_1} \, \ldots \, \mathcal{X}_{A_n}\delta)(g), \tag{66}$$

where $\mathcal{X}_A$ is the left-invariant vector field on group G representing the generator X_A whose explicit form is given in the appendix. Inserting this into (62) we obtain the corresponding function from $\mathcal{A}^\star_{\{e\}}$

$$F_{A_1\ldots A_n}(x) \; = \; (-1)^n(\mathcal{X}_{A_n} \, \ldots \, \mathcal{X}_{A_1}\omega)(g,x)|_{g=e}. \tag{67}$$

Here we used the fact that the operators $\mathcal{X}_A$ are anti-hermitian differential operators with respect to the group measure dg. From (62) it follows directly that

$$\begin{aligned} &(F_{A_1\ldots A_n} \star F_{B_1\ldots B_m})(x) \\ &= \; (-1)^{n+m}(\mathcal{X}_{A_n} \, \ldots \, \mathcal{X}_{A_1}\, \mathcal{X}_{B_m} \, \ldots \, \mathcal{X}_{B_1}\omega)(g,x)|_{g=e}. \end{aligned} \tag{68}$$

Equations (67) and (68) describe explicitly the homomorphism $\mathcal{U}(su(2,1)) \to \mathcal{A}^\star_{\{e\}}$.

Using exponential parametrization of the group element $g = e^{\xi^A X_A}$ formula for the symmetrized function (67) takes simple form:

$$\begin{aligned} F_{\{A_1\ldots A_n\}}(x) \; &= \; (-1)^n(\partial_{\xi_{A_1}} \, \ldots \, \partial_{\xi_{A_n}}\omega)(e^{\xi^A \hat{X}_A}x)|_{\xi=0} \\ &= \; (-1)^n \, \langle x| \, \hat{X}_{\{A_1} \, \ldots \, \hat{X}_{A_n\}}|x\rangle, \end{aligned} \tag{69}$$

where $\{\ldots\}$ means symmetrization of double indexes and $\xi = 0$ means the evaluation at $\xi_A = 0$ for $A_n = 1,\ldots,8$. Symmetrized functions form a basis of the algebra in question and symmetrized elements from the center of algebra correspond to Casimir operators. In the series of representation in question all Casimir operators are given in terms of a single operator $\hat{N}$ given in (103) which is represented by a constant function $N(x) = \langle x|\hat{N}|x\rangle = N$.

Example 1.: The function function $\omega(g,x)$. Let us calculate the function $\omega(g,x) = \omega_0(g_x^{-1}\, g\, g_x)$ explicitly. Taking g and g_x as in (17) and (59) we have to calculate the product of three matrices:

$$g_x^{-1}\, g\, g_x \; = \; \begin{pmatrix} C' & -S' \\ -S'^\dagger & C'' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C' & S \\ S^\dagger & C'' \end{pmatrix}.$$

Using equation (58) for $\omega_0(g)$ we obtain

$$\begin{aligned}\omega(g,x) &= \det[C'aC' + k''C'b\,(\bar{k}'_{12},\ \bar{k}'_{22})\sinh t - \sinh t\bar{k}''\begin{pmatrix}k'_{12}\\ k'_{22}\end{pmatrix}\ c\,C' \\ &- d\sinh^2 t\begin{pmatrix}k'_{12}\\ k'_{22}\end{pmatrix}(\bar{k}'_{12},\ \bar{k}'_{22})]^{-N} \qquad (70)\end{aligned}$$

After some basic calculations we have:

$$\begin{aligned}\omega(g,x) &= (\cosh t)^{-2N}\det[a + k''\tanh tb\,(\bar{k}'_{12},\ \bar{k}'_{22}) - \tanh t\bar{k}''\begin{pmatrix}k'_{12}\\ k'_{22}\end{pmatrix}\ c \\ &- d\tanh^2 t\begin{pmatrix}k'_{12}\\ k'_{22}\end{pmatrix}(\bar{k}'_{12},\ \bar{k}'_{22})]^{-N}. \qquad (71)\end{aligned}$$

We define the function ξ_A as the expectation value of the operator $\hat{X}_A$ between the coherent states as;

$$\xi_A(x) = \frac{1}{N}\langle x|\hat{X}_A|x\rangle = \frac{1}{N}\langle x_0|\hat{T}^\dagger(g_x)\,\hat{X}_A\hat{T}(g_x)|x_0\rangle, \qquad (72)$$

for $A = 1,\ldots,8$. Taking into account (44) we see that $\xi_A(x) = D_A^B(g_x)$, where $Ad_g^* = (D_A^B(g))$ is the matrix corresponding to the group action in co-adjoint algebra. Therefore it is sufficient to evaluate the coordinates at x_0: $\xi_A(x_0) = \frac{1}{N}\langle x_0|\hat{X}_A|x_0\rangle$.

The star product between these functions are defined as

$$\xi_A \star \xi_B = \frac{1}{N^2}\langle x|\hat{X}_A\hat{X}_B|x\rangle, \qquad (73)$$

or more explicitely:

$$(\xi_A \star \xi_B)(x) = \frac{1}{2N^2}\langle x|\{\hat{X}_A,\hat{X}_B\}|x\rangle + \frac{1}{2N^2}\langle x|[\hat{X}_A,\hat{X}_B]|x\rangle, \qquad (74)$$

where $\{\ldots\}$ denotes anti-commutator and $[\ldots]$ is commutator. Therefore, the second term is

$$\frac{1}{2N^2}\langle x|[\hat{X}_A,\hat{X}_B]|x\rangle = \frac{1}{2N}f_{A,B}^C\,\xi_C(x), \qquad (75)$$

where we used the definition of $\hat{X}_A$ and the short-hand notation for the commutator: $[X_A,X_B] = f_{A,B}^C X_C$. The first term is proportional to the symmetrized function $F_{\{A,B\}}$ and we can use (71):

$$\frac{1}{2N^2}\langle x|\{\hat{X}_A,\hat{X}_B\}|x\rangle = (1+A_N)\,\xi_A(x)\,\xi_B(x) + B_N\,\delta_{AB}, \qquad (76)$$

where we have a usual point-wise product of functions in the first term.

The coefficients A_N and B_N depend on the Bernoulli numbers coming from the Baker-Campbell-Hausdorff formula and are of order $1/N$.

So that we have:

$$(\xi_A \star \xi_B)(x) \;=\; (1 + A_N)\, \xi_A(x)\, \xi_B(x) \;+\; \frac{1}{2N}\, f^C_{A,B}\, \xi_C(x) \;+\; B_N\, \delta_{A,B}. \tag{77}$$

According to the Harish-Chandra imbedding theorem, we could always imbed the commutative maximal Hermitian symmetric space into the noncompact part of the Cartan subalgebra. We conjectured this is also true for the noncommutative Symmetric space whose coordinates is defined by ξ_A. For a basis of the $su(2,1)$ given in the appendix, the noncommutative coordinates correspond to the ones for $A = 4, 5, 6, 7$.

We see that the parameter of the non-commutativity is $\lambda_N = 1/N$. For $N \to \infty$ we recover the commutative product.

5 A quantum field theory model on $\hat{D}$

In this section we consider a quantum theory of scalar fields Φ on the deformed Bergman domain $\hat{D}$. The scalar field Φ could be written as a polynomial function of the noncommutative coordinates ξ_A (72). We consider only the case of discrete series of representation. The spectrum problem of the invariant Laplacian operator for the domain D is given in the appendix.

The Lagrangian could be defined as

$$S[\Phi] = \int d\mu(Z,\bar{Z})\{-\frac{1}{2}\Phi \star \Delta_N \Phi(Z,\bar{Z}) - \frac{1}{2}[\mu^2 + \xi R]\Phi^2(Z,\bar{Z})_\star - \lambda \Phi^4_\star\} \tag{78}$$

where $d\mu(Z,\bar{Z})$ is the invariant measure of the Bergman domain D given by (32), μ is the mass of the field Φ, $R = -(m+1)$ is the curvature scalar, see (29), and ξ is a numerical factor coupled with the curvature.

We expand the scalar field Φ in terms of the eigenfunction of the invariant Laplacian:

$$\Phi = \sum_l C_{N,l}\phi(N,l,Z), \tag{79}$$

where we summe over the discrete part of the spectrum. We use the path integral quantuization. The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral over fields Φ:

$$\langle F[\Phi] \rangle \;=\; \frac{\int D[\Phi]\, e^{-S[\Phi]}\, F[\Phi]}{\int D[\Phi]\, e^{-S[\Phi]}}, \tag{80}$$

where

$$\int D[\Phi] = \int_D \prod_{Z,\bar{Z}\in D} d\Phi(Z,\bar{Z}). \tag{81}$$

The propagator in the Fourier space reads :

$$< \Phi, \Phi >= \frac{1}{\mu^2 - (m+1)\xi + \frac{1}{4}((N-2)^2 + \lambda^2)} \tag{82}$$

Now we consider the discrete sprectrum $\lambda = i(N-2-2l)$ which contains $l(l+2-N)$ points, $l = 0, 1, \cdots, \left[\frac{N-2}{2}\right]$. For the $m = 2$ case the propagator reads:

$$\frac{1}{l(N-2) - l^2 + \mu^2 - 3\xi}. \tag{83}$$

From (77) we find that the first order noncommuative correction to the product of the noncommutative coordinates is $1/N$, so for simplicity we suppose the lowest order noncommutative correction to the $\Phi^4_\star$ term is is also $1/N$. The precise formula for the noncummutative correction should be of order $1/N$ plus higher orders.

Now we calculate the amplitude for the tadpole graph. For simplicity we omit the inessential factor ξ, we have

$$\begin{aligned} G_2 &= \sum_{N=3}^{\infty} \sum_{l=0}^{[(N-2)/2]} \frac{1}{N} \frac{1}{l(N-2) - l^2 + \mu^2} \\ &= \sum_{N=3}^{\infty} \sum_{l=0}^{[(N-2)/2]} \frac{1}{N} \frac{1}{\mu^2 + (\frac{N-2}{2})^2 - (l - \frac{N-2}{2})^2} \end{aligned} \tag{84}$$

We consider the massless case $\mu = 0$. Summing over the indices l, the amplitude of the tadpole graph reads:

$$G_2 = \sum_{N=3}^{\Lambda} \frac{1}{N} \frac{1}{N-2} [\ln(N-2) - \ln \epsilon]. \tag{85}$$

where $0 < \epsilon \ll 1$ is the infrared regulator of l. We find that G_2 diverges when $N << \infty$, which is a kind of infrared divergence. If we allow N to be arbitrarily big and assume that $\epsilon \sim \frac{1}{\Lambda}$ without losing generality, then the 1 loop correction reads:

$$G_2 = \lambda(1 - \frac{1}{\Lambda} \ln \Lambda - \frac{1}{\Lambda}) < \infty, \tag{86}$$

which is not divergent. So that we could kill the infrared divergence by taking the ultraviolet limit. Note that in the massless case in order that the propagator being positive, the coupling constant ξ should also vanish. Similarly we could find that the amplitude of the tadpole graph for the massive model is also finite.

6 Appendix A, the Lie algebra $su(2,1)$

The Lie algebra $\mathbf{g} = su(2,1)$ is formed by 3×3 complex matrices X satisfying

$$X^\dagger \Gamma \ + \ \Gamma X \ = \ 0. \tag{87}$$

It has 8 independent generators that form an orthogonal complete basis. We could choose them as:

$$X_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, X_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, X_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \tag{88}$$

Under this basis the infinitesimal generator or the Killing verctors could be written as:

$$\begin{aligned}
\hat{X}_1 &= i(z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1}),\ \hat{X}_2 = -z_1\frac{\partial}{\partial z_2} + z_2\frac{\partial}{\partial z_1},\ \hat{X}_3 = i(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}),\\
\hat{X}_4 &= i\frac{\partial}{\partial z_1} + iz_1^2\frac{\partial}{\partial z_1} + iz_1z_2\frac{\partial}{\partial z_2},\ \hat{X}_5 = \frac{\partial}{\partial z_1} - z_1^2\frac{\partial}{\partial z_1} - z_1z_2\frac{\partial}{\partial z_2},\\
\hat{X}_6 &= i\frac{\partial}{\partial z_2} + iz_1z_2\frac{\partial}{\partial z_1} + iz_2^2\frac{\partial}{\partial z_2},\ \hat{X}_7 = \frac{\partial}{\partial z_2} - z_1z_2\frac{\partial}{\partial z_1} - z_2^2\frac{\partial}{\partial z_2},\\
\hat{X}_8 &= -\frac{i}{\sqrt{3}}(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}) + \frac{2n}{\sqrt{3}}i.
\end{aligned} \tag{89}$$

Where $Z = (z_1, z_2)^\dagger$ is the coordinate of the coset space D and n is a number to characterize the representation. More explictly, $n = a + ib$ where $a, b \in R$ for the principal series, $n \in N$ for the discrete series and $n \in (-1/2, 0)$ for the supplementary series of representation.

7 Appendix B, the harmonic oscillator representation for $SU(m,n)$

In this section we realize the maximal degenerate discrete series of representations for the $SU(m,n)$ group in terms of harmonic oscillators.

We introduce a $m \times n$ matrix $\hat{Z} = (\hat{z}_{a\alpha})$, $a = 1, \ldots, m$, $\alpha = 1, \ldots, n$, of bosonic oscillators that act in Fock space and satisfy the following commutation relations:

$$[\hat{z}_{a\alpha}, \hat{z}^\dagger_{b\beta}] = \Gamma_{ab}\,\delta_{\alpha\beta}, \quad a, b = 1, 2, \tag{90}$$

$$[\hat{z}_{a,\alpha}, \hat{z}_{b\beta}] = [\hat{z}^\dagger_{a,\alpha}, \hat{z}^\dagger_{b\beta}] = 0\,, \tag{91}$$

where Γ is a $(m+n) \times (m+n)$ matrix defined in (2). It can be easily seen that these commutation relations are invariant under transformations:

$$\hat{Z} \mapsto g\,\hat{Z}, \quad \hat{Z}^\dagger \mapsto \hat{Z}^\dagger\,g^\dagger, \quad g \in SU(m,n). \tag{92}$$

From (90) it follows that the upper $m \times n$ block in $\hat{Z}$ corresponds to annihilation operators whereas the lower $m \times m$ block is formed from creation operators. So we rewrite the matrix $\hat{Z}$ as

$$\hat{Z} = \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix} \tag{93}$$

such that

$$
\begin{aligned}
[\hat a_{a\alpha}, \hat a^\dagger_{b\beta}] \;&=\; \delta_{ab}\delta_{\alpha\beta}, \quad a, b = 1, \ldots, m, \;\; \alpha, \beta = 1, \ldots, n, \\
[\hat b_{a\alpha}, \hat b^\dagger_{b\beta}] \;&=\; \delta_{ab}\,\delta_{\alpha\beta}, \quad a, b, \alpha, \beta = 1, \ldots, n.
\end{aligned} \tag{94}
$$

All other commutation relations between annihilation (creation) operators vanish. The Fock space $\mathcal{F}$ in question is generated from a normalized vacuum state $|0\rangle$, satisfying $\hat a_\alpha\,|0\rangle \;=\; \hat b\,|0\rangle \;=\; 0$, by repeated actions of creation operators:

$$
|m,\, n\rangle \;=\; \prod_{a\alpha, b\beta} \frac{(\hat a^\dagger_{a\alpha})^{m_{a\alpha}}\,(\hat b^\dagger_{b\beta})^{n_{b\beta}}}{\sqrt{m_{a\alpha}!\, n_{b\beta}!}}\,|0\rangle\,. \tag{95}
$$

Here, $m \;=\; (m_{a\alpha})$ and $n \;=\; (n_{b\beta})$ are matrices of non-negative integers and the range of indexes a, α and b, β have been indicated in (94). We shall use the terminology that the state $|m,\, n\rangle$ contains $M = \sum m_{a\alpha}$ particles a and $N = \sum n_{b\beta}$ particles b.

The Lie algebra $su(m, n)$ acting in Fock space can be realized in terms of oscillators. To any basis $X^{AB} \in \mathbf{g}$, where $A, B = 1, 2 \cdots (m+n)^2 - 1$ we assign the *anti-hermitian* operator

$$
\hat{X^{AB}} \;=\; -\mathrm{tr}\,(\hat Z^\dagger \Gamma X^{AB} \hat Z) \;=\; -\hat z^\dagger_{a\alpha}\,\Gamma_{ac}\,X^{AB}_{cb}\,\hat z_{b\alpha}\,, \tag{96}
$$

with $\hat Z^\dagger$ and $\hat Z$ given in (90) and (93) in terms of oscillators.

Using commutation relations for annihilation and creation operators the commutator of operators $\hat X \;=\; \hat Z^\dagger \Gamma X \hat Z$ and $\hat Y \;=\; \hat Z^\dagger \Gamma Y \hat Z$ can be easily calculated:

$$
[\hat X, \hat Y] \;=\; [\hat Z^\dagger \Gamma X \hat Z, \hat Z^\dagger \Gamma Y \hat Z] \;=\; -\hat Z^\dagger\,\Gamma\,[X,\; Y]\,\hat Z\,. \tag{97}
$$

It can be easily seen that the anti-hermitian operators $\hat X_A$ satisfy the $su(m, n)$ commutation relations in Fock space. The assignment

$$
g \;=\; e^{\xi^A X_A} \;\in\; SU(m, n) \;\Rightarrow\; \hat T(g) \;=\; e^{\xi^A \hat X_A} \tag{98}
$$

then defines the unitary $SU(m, n)$ representation in the Fock space.

The adjoint action of $\hat T(g)$ on operators reproduces the left-action (92). In terms of a and b-oscillators in block-matrix notation this can be rewritten as

$$
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad
\begin{aligned}
&\hat T(g)\,\hat a\,\hat T^\dagger(g) \;=\; a\,\hat a + b\,\hat b^\dagger, \quad \hat T(g)\,\hat a^\dagger \hat T^\dagger(g) \;=\; \hat a^\dagger\, a^\dagger + \hat b\, b^\dagger, \\
&\hat T(g)\,\hat b^\dagger\,\hat T^\dagger(g) \;=\; d\,\hat b^\dagger + c\,\hat a, \quad \hat T(g)\,\hat b\,\hat T^\dagger(g) \;=\; \hat b\, d^\dagger + \hat a^\dagger\, c^\dagger.
\end{aligned} \tag{99}
$$

Since any $g \;\in\; SU(m, n)$ possesses Cartan decomposition (17) we shall discuss separately rotations and special boosts.

For any rotation $k \;\in\; \mathbf{k} \;=\; s(u(m) \oplus u(n))$ we obtain the mixing of annihilation and creation operators of the same type:

$$
k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} : \quad
\begin{aligned}
&\hat T(k)\,\hat a\,\hat T^\dagger(k) \;=\; k_1\,\hat a, \quad \hat T(k)\,\hat a^\dagger \hat T^\dagger(k) \;=\; \hat a^\dagger\, k_2, \\
&\hat T(k)\,\hat b^\dagger\,\hat T^\dagger(k) \;=\; k_2\,\hat b^\dagger, \quad \hat T(k)\,\hat b\,\hat T^\dagger(k) \;=\; \hat b\, k_2^\dagger.
\end{aligned} \tag{100}
$$

where $\hat a \;=\; (\hat a_{a\alpha})$, $\hat a^\dagger \;=\; (\hat a^\dagger_{a\alpha})$, $\hat b \;=\; (\hat b_{b\beta})$, $\hat b^\dagger \;=\; (\hat b^\dagger_{b\beta})$ denote matrices consisting from the corresponding oscillator operators.

However, for the special boost δ_Λ given in (10) we obtain Bogolyubov transformations. In matrix notation the Bogolyubov transformation takes the form:

$$\begin{aligned} \hat{T}(\delta_\Lambda)\,\hat{Z}\,\hat{T}^\dagger(\delta_\Lambda) &= \delta_\Lambda\,\hat{Z}, \\ \hat{T}(\delta_\Lambda)\,\hat{Z}^\dagger\,\hat{T}^\dagger(\delta_\Lambda) &= \hat{Z}^\dagger\,\delta_\Lambda. \end{aligned} \tag{101}$$

All other boosts can be obtained from the special boosts by rotations.

So as the same in the $SU(2,1)$ case, the action of *rotation* generators results in a replacement of some a-particle by an other a-particle and by replacement of b-particle by other b-particle and the action of *boost* generators results in a creation of some ab-pair of particles or in a destruction of some other ab pair.

Consider the lowering and rising operators that annihilate and create some $a_{a\alpha}b_{b\beta}$ pair:

$$T^-_{a\alpha,b\beta} \;=\; \hat{a}_{a\alpha}\,\hat{b}_{b\beta}, \quad T^+_{a\alpha,b\beta} = (T^-_{a\alpha,b\beta})^\dagger \;=\; \hat{b}^\dagger_{b\beta}\hat{a}^\dagger_{a\alpha}. \tag{102}$$

Any boost can be uniquely expressed as complex combinations of lowering and rising operators.

It follows from (102) that the operator

$$\hat{N} \equiv\; \hat{N}_{\hat{b}} - \hat{N}_{\hat{a}} \;=\; \sum_{b\beta} \hat{b}^\dagger_{b\beta}\hat{b}_{b\beta} \;-\; \sum_{a\alpha} \hat{a}^\dagger_{a\alpha}\hat{a}_{a\alpha} \;=\; -\mathrm{tr}\,(\hat{Z}^\dagger \Gamma \hat{Z}) \;-\; n^2 \tag{103}$$

commutes with all generators $\hat{X}_A$.

We start to construct the representation space $\mathcal{F}_N$ from a distinguished normalized state containing lowest number of particles:

$$|x_0\rangle = c_N^{-1}\,(\det b^\dagger)^N\,|0\rangle\,. \tag{104}$$

where c_N is normalization coefficient and N is the natural number that specifies the representation: $\hat{N}\,|x_0\rangle \;=\; N\,|x_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in (102): such states contain besides nN b-particles a finite number of ab pairs.

8 Appendix C: The invariant Laplacian for the commutative Bergman domain D

The invariant Laplacian Δ_N on the Bergman domain $D = SU(m,1)/U(m)$ reads ([16][17]):

$$\Delta_N = (1-|Z|^2)(\sum_{i=1}^m \frac{\partial^2}{\partial \bar{z}_i \partial z_i} - \bar{R}R - N\bar{R}) \tag{105}$$

where $R \;=\; \sum_1^m z_i \partial/\partial z_i$ is the first-order differential operator. Here we consider only the radial part of the invariant Laplacian and restrict ourselves to the discrete set of eigenfunctions.

A key property of Δ_N is that it is invariant under the action of the representation operator.

$$T_g \Delta_N f(Z) = \Delta_N T_g f(Z) \tag{106}$$

The interested could find in ([15]) the explicit form of Δ_N for arbitrary type I Cartan domain.

The eigenfunction of Δ_N reads ([16][17]):

$$\phi_\lambda(Z) = (1-|Z|^2)^{(-N+2-i\lambda)/2} F(\frac{N+2-i\lambda}{2},\ \frac{-N+2-i\lambda}{2};\ m;\ |Z|^2) \tag{107}$$

where $F(\frac{N+2-i\lambda}{2},\ \frac{-N+2-i\lambda}{2};\ m;\ |Z|^2)$ is the hypergeometric function, and λ is an complex number which is the dual of the complex roots:

$$\lambda = \alpha(\mathbf{g}_c), \tag{108}$$

and the eigenvalue read:

$$-\frac{1}{4}((N-2)^2 + \lambda^2). \tag{109}$$

The discrete series of representation corresponds to

$$\lambda = i(N-2-2l) \tag{110}$$

where $l = 0, 1, \cdots, \left[\frac{N-2}{2}\right]$ we have discrete spectrum, which contains finite many points:

$$l(l+2-N), \quad l = 0, 1, \cdots, [\frac{N-2}{2}]. \tag{111}$$

Ackonledgements This paper is based on the joint work and interesting discussions with Harald Grosse and Peter Presnajder. Their suggestions are very important to the completion of this paper. The author is also grateful to Michel Dubois-Violette, Xiaonan Ma, Vincent Rivasseau, Robin Zegers and Genkai Zhang for useful discussions.

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